# THERE IS NO CONTINUOUS PROJECTION FROM THE MEASURABLE FUNCTIONS ON THE SQUARE ONTO THE MEASURABLE FUNCTIONS ON THE INTERVAL[†]

BY

I. D. BERG, N. T. PECK AND H. PORTA

ABSTRACT

It is proved that the space of functions constant on vertical lines is not complemented in the space of all measurable functions on the unit square (with the topology of convergence in measure). The analogous result is proved for the space of all measurable functions on the product of two probability spaces, one of which is atomless.

A. Pelczynski has raised the interesting question of whether there is a continuous projection from the measurable (real or complex-valued) functions on the unit square onto those on the unit interval; he conjectured that there is none. We establish that conjecture and also the generalization to products of probability spaces.

We note that, as measure spaces, the unit square and the unit interval are isometric and therefore $S^2$, the space of (equivalence classes of) measurable functions on the square, is isometric to $S$, the space of measurable functions on the interval. Hence, identifying the space of measurable functions on the square which are constant on vertical lines with $S$, we will obtain (as an easy corollary) an uncomplemented subspace of $S^2$ which is isometric to $S^2$.

We also note, for example, that the Banach space $L_\infty$ of essentially bounded functions on the square can be continuously projected onto the essentially bounded

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functions constant on vertical lines by integrating with respect to the vertical coordinate. However, on $L_\infty$ integration is clearly discontinuous in the topology of convergence in measure.

Recall that the topology of convergence in measure on $S^2$ is induced by the metric

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}\,d\mu,$$

where $\mu$ is two-dimensional Lebesgue measure. We will simplify the exposition and notation by freely disregarding sets of measure zero; we will be dealing with countable processes only and therefore will encounter no difficulties by calling sets of measure zero "empty". At two places in the proof of Theorem 1 there are statements whose proofs are more or less routine. In order not to interrupt the exposition, we number these with superscripts and postpone the proofs to the end.

THEOREM 1. *There cannot exist a continuous projection $P: S^2 \to S^2$ whose image consists of functions constant on vertical lines and which is the identity on such functions. That is, there is no continuous projection from the measurable functions on the square onto the measurable functions on the interval.*

PROOF. Let $P$ denote the supposed projection. Arrange the set of rational subsquares of the square in a sequence $\{W_i\}$, and define $\hat{P}$, a set-valued function on the measurable subsets of the square, by

$$\hat{P}(E) = \bigcup_{i=1}^{\infty} \text{Support } P[\chi(E \cap W_i)].$$

(As usual, $\chi(A)$ denotes the characteristic function of $A$.) Thus, $\hat{P}(E)$ is the union of the supports of the images of a sequence of functions which span a dense subspace of the space of functions supported in $E$. It follows that if Support $f \subset E$, $Pf$ vanishes off $\hat{P}(E)$. The use of the $\{W_i\}$ is just to prevent sets of measure 0 from accumulating over an uncountable index set. We could use a more economical choice of sets than the $\{W_i\}$ in our proof, and indeed this is required in the general case treated in Theorem 2; however, in the present case the expository convenience compensates us for the mathematical redundancy.

We now note that if we can find a set $K$ of measure $a$ so that $\mu[\hat{P}(K)] = b$, then we can find a function $f$ whose support is included in $K$ and hence is of measure at most $a$ and such that $\mu$ (Support $Pf$) $= b$. (Recall that $\mu$ is two-dimensional

Lebesgue measure.) Indeed, $\sum_{i=1}^{\infty} \alpha_i \chi(K \cap W_i)$ will have that property when each $\alpha_n$ is chosen so that $\alpha_n P[\chi(K \cap W_n)] \neq \sum_{i=1}^{n-1} \alpha_i P[\chi(K \cap W_i)]$ holds everywhere on the supports of these functions, except for a set of measure zero. The $\alpha_n$ are chosen as rapidly decreasing as is necessary to guarantee convergence in measure of $\sum_{i=1}^{\infty} \alpha_i(\chi(K \cap W_i)$; they are also chosen so that $P \sum_{i=1}^{\infty} \alpha_i \chi(K \cap W_i)$ is never zero on $\hat{P}(K)$.[(1)]

Hence if we can find a sequence $\{K_n\}$ of measurable sets such that $\mu(K_n) \underset{n}{\to} 0$ but $\mu[\hat{P}(K_n)] \not\to 0$, we can by the above remark find a sequence $f_n \to 0$ in $S^2$ but such that $Pf_n \not\to 0$ in $S^2$. Indeed, associate to each $K_n$ a function $f_n$ as described above. Then $\{\beta_n f_n\}$, for sufficiently rapidly growing scalars $\{\beta_n\}$, will provide such a sequence. (Note that for a fixed measurable function $g$,

$$\lim_{m \to \infty} \int \frac{|mg|}{1+|mg|} d\mu = \mu \text{ (Support } g).)$$

We now come to the main part of the proof, the construction of a set $K_n$ of measure $1/n$ but such that $\mu[\hat{P}(K_n)] = 1$. A rough description of $K_n$ may be useful. The square is partitioned into $n$ horizontal strips $F_j$, $1 \leqq j \leqq n$, each of height $1/n$. For each of the first $n-1$ strips, say the $j$th, an "invariant" vertical strip $F_j^\infty$ is found, invariant in the sense that $\hat{P}(F_j^\infty \cap F_j) \supset F_j^\infty$. The strips $F_j^\infty$ are pairwise disjoint. Most important, our construction of the invariant strips $F_j^\infty$ guarantees that $F_n^\infty = I^2 \setminus \bigcup_{i=1}^{n-1} F_j^\infty$ is itself an invariant strip (where $I^2$ denotes the unit square). The set $K_n$ is then defined to be $\bigcup_{j=1}^{n} (F_j^\infty \cap F_j)$.

We first describe a construction which associates to each measurable rectangle a countable partition of the vertical strip whose base coincides with that of the rectangle. Let $R$ be a rectangle and let $T_0$ be the vertical strip with the same base. Consider the nested set of substrips of $T_0$:

$$T_1 = \hat{P}(R) \cap T_0$$

$$\vdots$$

$$T_{n+1} = \hat{P}(T_n \cap R) \cap T_0.$$

Set $R^\sigma = \bigcap_{i=0}^{\infty} T_i$. Then the strips $\{R^i = T_i \setminus T_{i+1}\}$, $i = 0, 1, \cdots$ together with $R^\sigma$ form the countable partition described. A glance at the construction makes it clear that $\hat{P}(R^i \cap R) \cap R^i = \varnothing$ for $i = 0, 1, 2, \cdots$, and it is not hard to establish the invariance property of $R^\sigma$; even more, $\hat{P}(R^\sigma \cap R) \cap T_0 = R^\sigma$.[(2)]

Note that a union of $R^i$ will not in general have either of the above desirable properties.

We now construct $K_n$. First we partition the unit square into $n$ horizontal strips $F_1, \cdots, F_n$ each of height $1/n$. The square is divided into a countable number of vertical strips by applying the construction just described to $F_1$ (that is, $F_1$ plays the role of $R$) to get $F_1^0, \cdots, F_1^n, \cdots, F_1^\sigma$. We set $F_1^\infty = F_1^\sigma$. Now apply the same construction to each of $F_1^0 \cap F_2$, $F_1^1 \cap F_2, \cdots$ (but not to $F_1^\infty$) and get a new countable partition, each vertical strip a substrip of one of the strips constructed at the preceding stage. We set $F_2^\infty = \bigcup_{j=0}^\infty (F_1{}^j \cap F_2)^\sigma$ and arrange the remaining strips into a sequence $F_2^i$, $i = 0, 1, 2, \cdots$. We now have $\hat{P}(F_2^\infty \cap F_2) \supset F_2^\infty$, (inclusion rather than equality is claimed here because we have no reason to assume that $\hat{P}[(F_1^j \cap F_2)^\sigma] \subset F_1^j$), while $\hat{P}(F_1^\infty \cap F_1) = F_1^\infty$ and $\hat{P}(F_2^i \cap F_2) \cap F_2^i = \emptyset$, for $i = 0, 1, 2, \ldots$. Then $F_1^\infty$, $F_2^\infty$, $\{F_2^i\}_{i=0}^\infty$ form a partition of the square.

We next apply the same construction to $\{F_2^i \cap F_3\}$ and continue until we have dealt with $F_{n-1}$ and have thus obtained $F_1^\infty, F_2^\infty, \cdots, F_{n-1}^\infty$, $\{F_{n-1}^i\}_{i=0}^\infty$ forming a countable partition of the square. Figure 1 schematically illustrates the construction for the case $n = 3$.

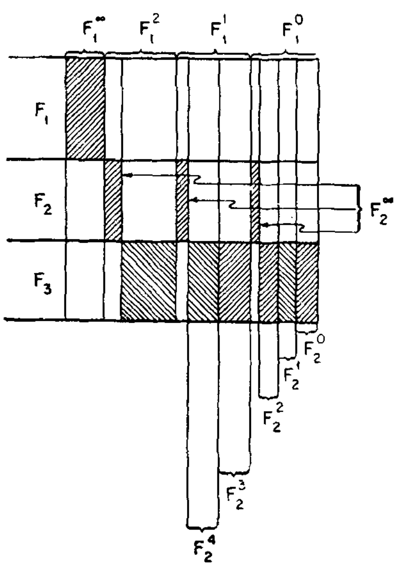


Fig. 1. Because of the complexity of the diagram we have made no indication of the infinite collections of sets that occur and labeled only a few of the sets that are indicated.

Now we use the fact that $P$ is a projection by noting that $P\chi(F_{n-1}^i) = \chi(F_{n-1}^i)$ and so $\hat{P}(F_{n-1}^i) \supset F_{n-1}^i$. But we observe that for $j = 1, \cdots, n-1$, we have

$\hat{P}(F^i_{n-1} \cap F_j) \cap F^i_{n-1} = \emptyset$ because $F^i_{n-1}$ was so constructed for $F_{n-1}$ and is a substrip of a strip satisfying this relationship for $j < n-1$. Hence $\hat{P}(F^i_{n-1} \cap F_n) \supset F^i_{n-1}$ and so $F^\infty_n = \bigcup^\infty_{i=0} F^i_{n-1}$, the union of all the strips that have not already appeared as $F^\infty_j$ for $j \leqq n-1$, is itself an invariant strip for $F_n$.

We now have the following collection of $n$ measurable rectangles, each of height $1/n$ lying in disjoint vertical strips: $F_1 \cap F^\infty_1$, $F_2 \cap F^\infty_2, \cdots$, $F_n \cap F^\infty_n$. Recall that $F^\infty_i$ is a vertical strip whose measure we do not know, but $F_i$ is a horizontal strip of measure $1/n$. We note that $\hat{P}(F_i \cap F^\infty_i) \supset F^\infty_i$. Then $\mu(\bigcup^n_{i=1} (F_i \cap F^\infty_i)) = 1/n$ but $\hat{P}[\bigcup^n_{i=1} (F_i \cap F^\infty_i)] = \bigcup^n_{i=1} F^\infty_i$, the entire square.

If we repeat this construction for each $n$ and let $K_n = \bigcup^n_{i=1} (F_i \cap F^\infty_i)$ (a different set of $\{F_i\}$ for each $n$) we see that $K_n$ is a sequence of sets whose measure approaches zero, but with the property that $\mu[\hat{P}(K_n)] = 1$. This is the sequence we undertook to construct and whose existence contradicts the assumed continuity of $P$. This completes the proof of Theorem 1.

Theorem 1 has the following generalization:

THEOREM 2. *Let $X_1, X_2$ be probability spaces (measure spaces with a positive measure of total mass* 1). *Let $S'$ denote the space of measurable functions on $X_1 \times X_2$ with the topology of convergence in measure. Then if and only if $X_2$ has an atom can there exist a continuous projection $P: S' \to S'$ whose image consists of functions constant for fixed $X_1$ coordinates and which is the identity on such functions. That is, there is a projection from the measurable functions on $X_1 \times X_2$ onto the measurable functions on $X_1$ if and only if $X_2$ has an atom.*

PROOF. The proof is the same as that of Theorem 1 with one major exception if the measure spaces are nonseparable, we must search for a replacement for the sequence $\{W_i\}$. However, we once again break up $X_1 \times X_2$ into the $n$ "horizontal" strips $F_1, \cdots, F_n$ each of measure $1/n$ (by virtue of $X_2$ being atomless) and we consider, for example, $A_n$, the countable algebra generated by the supports of the successive applications of $P$ to $\chi(F_i)$, and the sets that so arise. Replace $\hat{P}$ by $\tilde{P}$ defined by $\tilde{P}(E) = \bigcup_{A \in A_n}$Support $P\chi(E \cap A)$ and proceed as before. This algebra is big enough to allow us to achieve $\tilde{P}(E \setminus \tilde{P}(E)) \cap E \setminus \tilde{P}(E) = \emptyset$ for any set $E$ arising in our construction. We conclude the proof of Theorem 2 by noting that if $X_2$ has an atom at $y_0$ then $Pf(x,y) = f(x, y_0)$ is a suitable $P$.

**Proofs of superscripted statements**

PROOF OF 1. We first treat the case of finitely many functions. We assert that if $h_1, \cdots, h_n$ are in $S^2$ and $A_i =$ Support $h_i$, then there is a linear combination of $h_1, \cdots h_n$ whose support is $\bigcup_{i=1}^n A_i$ (except for a set of measure 0). By induction, it is sufficient to prove this when $n = 2$. If $X$ is any uncountable set of real numbers, for each $\alpha \in X$ let $Z_\alpha$ be the intersection of the zero set of $h_1 + \alpha h_2$ and Support $h_2$. The sets $Z_\alpha$, $\alpha \in X$, are pairwise disjoint, so for at least one $\beta \in X$, $\mu(Z_\beta) = 0$. The desired linear combination is then $h_1 + \beta h_2$.

Now suppose that $h_i = Pg_i$ and it is desired to find a series $\sum_{i=1}^\infty \alpha_i g_i$ convergent in measure such that Support $[\sum_{i=1}^\infty \alpha_i h_i] = \bigcup_{i=1}^\infty$ Support $h_i$. If $\alpha_1, \cdots, \alpha_n$ have been chosen (by the above) so that Support $[\sum_{i=1}^n \alpha_i h_i] = \bigcup_{i=1}^n$ Support $h_i$, then there are positive numbers $\varepsilon_n$ and $\delta_n$ such that $|\sum_{i=1}^n \alpha_i h_i| > \varepsilon_n$ on $\bigcup_{i=1}^n$ Support $h_i$ except for a set of measure $\delta_n$, with $\delta_n \underset{n}{\rightarrow} 0$. Then $\alpha_{n+1}, \alpha_{n+2}, \cdots$, each of which may be chosen from all but a countable set of scalars, may be chosen to satisfy $\sum_{i=n+1}^\infty \alpha_i h_i| < \varepsilon_n/2$ on $\bigcup_{i=1}^n$ Support $h_i$ except for a set of measure $\delta_n$. The $\alpha_n$ can also be chosen to converge to zero rapidly enough so that $\sum_{i=1}^\infty \alpha_i g_i$ is convergent in measure. We leave further details to the reader.

PROOF OF 2. We note by virtue of (1) that continuity of $P$ implies that for any sequence $E_n$, $\mu(E_n) \rightarrow 0 \Rightarrow \mu[\hat{P}(E_n)] \rightarrow 0$. Since $T_i \supset R^\sigma$ and $\mu(T_i \setminus R^\sigma) \rightarrow 0$, we see that $\mu[\hat{P}((T_i \setminus R^\sigma) \cap R)] \rightarrow 0$. But $\hat{P}(T_i \cap R) \supset R^\sigma$. Therefore $\hat{P}(R^\sigma \cap R) \supset R^\sigma$. The reverse inclusion is obvious.

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS, U.S.A.